Dynamics of Robertson-Walker spacetimes with diffusion (arXiv:1409.4400 [gr-qc])

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CAMGSD

Introduction

- Diffusion is the cause for several physical processes
  - Heat conduction
  - Brownian motion
- At the microscopic level diffusion is due to random collisions between the particles of the system with those of the background substance
  - Stochastic differential equations
- At the macroscopic scale, random effects are averaged, and diffusion is described by an effective and deterministic theory
  - Relativistic kinetic Fokker-Planck equation for distribution function f.
- New cosmological model in which the fluid particles undergo diffusion in a scalar field, representing the dark energy content of the Universe.
  - S. Calogero: A kinetic theory of diffusion in general relativity with cosmological scalar field. JCAP 11/2011, 016 (2011)
  - S. Calogero: Cosmological models with fluid matter undergoing velocity diffusion. J. Geom. Phys. **62**, 2208–2213 (2012)

Diffusion in General Relativity

- The energy-momentum tensor for a perfect fluid

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu} + p(g^{\mu\nu} + u^{\mu} u^{\nu})$$

with linear equation of state

$$p = (\gamma - 1)\,\rho \quad \text{and} \quad \frac{2}{3} < \gamma < 2.$$

- Energy current

$$J^{\mu} = n u^{\mu}$$

- For matter undergoing velocity diffusion

$$\nabla_{\mu} T^{\mu\nu} = \sigma J^{\nu}$$

$$\nabla_{\mu} J^{\mu} = 0$$

where σ is the diffusion constant and measures the average energy transferred per unit time from the background substance to a fluid particle.

- Einstein equations

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$$

  - In presence of diffusion $T_{\mu\nu}$ is not divergence-free. Incompatibility with the twice contracted Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$.
  - Add a matter field which interacts with the fluid particles restoring the local conservation of energy
  - The new matter field plays the role of a background medium in which particles undergo diffusion
  - The simplest model for this medium is a vaccum-energy described by a cosmological scalar field (varying Λ)

$$G_{\mu\nu} + \phi g_{\mu\nu} = T_{\mu\nu}$$

  - The diffusion equation is

$$\nabla_\mu \phi = \sigma J_\mu$$

  - When $\sigma = 0$ the model reduces to the Einstein-Euler system with cosmological constant Λ.

- Our main goal is to give a complete characterization of solutions in the Robertson-Walker geometry.

Robertson-Walker spacetimes

- In comoving coordinates (t, r, θ, φ) the metric reads

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right], \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2,$$

where $k = 0, \pm 1$ is the curvature parameter.

- The equations for the scale factor $a(t)$, the energy density $\rho(t)$ of the fluid and the cosmological scalar field $\phi(t)$ are

$$\begin{aligned}
\dot{a} &= Ha, \\
\dot{\rho} &= -3\gamma H\rho - \dot{\phi}, \\
\dot{\phi} &= -\sigma n_0\left(\frac{a_0}{a(t)}\right)^3, \\
\dot{H} &= \frac{1}{3}\left[\phi - \left(\frac{3}{2}\gamma - 1\right)\rho\right] - H^2, \\
H^2 &= \frac{1}{3}(\rho + \phi) - \frac{k}{a(t)^2},
\end{aligned}$$

where $a_0 > 0$, $n_0 > 0$ are the values of the scale factor and the particle density at the time $t = 0$.

- The deceleration parameter is defined via $q = -1 - \frac{\dot{H}}{H^2}$. A solution is accelerating iff $q < 0$.

Classification of general solutions

- Classification of general solutions based on their asymptotic behavior toward the past and future time directions.
  - A solution is of type **A** if it becomes singular at some finite time in the past, while in the future it is singularity free and asymptotically de-Sitter.
  - A solution is said to be of type **B** if it can be matched to a de-Sitter solution at some finite time in the past, while in the future it is singularity free and asymptotically de-Sitter.
  - Solutions of type **C** are those which can be matched to a vacuum solution at some finite time in the past and which become singular at some finite time in the future.
  - Finally, a solution of type **D** is a solution that becomes singular at finite time in both time directions.
- **MAIN RESULT**:
  Let $\mathcal{I}_k$ be the 3-dimensional manifold of the initial data. For all $k = 0, \pm 1$, there exists four disjoint three-dimensional submanifolds of initial data $\mathcal{A}_k \subset \mathcal{I}_k$, $\mathcal{B}_k \subset \mathcal{I}_k$, $\mathcal{C}_k \subset \mathcal{I}_k$, $\mathcal{D}_k \subset \mathcal{I}_k$ such that if the initial data belong to $\mathcal{A}_k$, the corresponding solution is of type **A**, if the initial data belong to $\mathcal{B}_k$, the corresponding solution is of type **B**, etc...
  - solutions which are not launched by initial data in the set $\mathcal{A}_k \cup \mathcal{B}_k \cup \mathcal{C}_k \cup \mathcal{D}_k$ are atypical, that is to say, they correspond to initial data forming a two-dimensional submanifold of $\mathcal{I}_k$.

Diffusion solutions

- When the scale factor $a(t)$ depends linearly on time, we find the explicit solution:

$$a(t) = a_0 + \delta_k t,$$
$$\phi(t) = \frac{3\beta}{2\delta_k} a(t)^{-2},$$
$$\rho(t) = \frac{3\beta}{\delta_k(3\gamma - 2)} a(t)^{-2},$$

  where

$$\beta = \frac{\sigma n_0 a_0^3}{3}$$

  and δ_k is the real solution of the polynomial equation

$$\delta^3 + k\delta - \frac{3\beta\gamma}{2(3\gamma - 2)} = 0.$$

- Note that $\delta_k > 0$, for all $k = 0, \pm 1$.

- For $k = 0$

$$a(t) = a_0 + \left(\frac{3\beta\gamma}{2(3\gamma-2)}\right)^{1/3} t,$$

$$\phi(t) = \left(\sqrt{\frac{3\gamma-2}{\gamma}}\frac{3\beta}{2}\right)^{2/3} a(t)^{-2},$$

$$\rho(t) = \left(\frac{2}{\gamma}\right)^{1/3}\left(\frac{3\beta}{3\gamma-2}\right) a(t)^{-2}.$$

- For $\sigma = 0$
  - For $k = 0$ reduces to the Minkowski spacetime
  - For $k = -1$ reduces to the Milne spacetime
  - For $k = 1$ there is no diffusion-free analogue

- All solutions are singularity free and forever expanding in the future, while in the past they become singular at the time $t_- = -a_0/\delta_k$.

- The future expansion takes place at a constant rate, i.e. $q = 0$.

- The singularity at $t = t_-$ is a curvature singularity. The Ricci scalar of the solution blows up,

$$R = 4\phi + (4-3\gamma)\rho = \frac{9\beta\gamma}{\delta_k(3\gamma-2)} a(t)^{-2} \to +\infty, \quad \text{as} \quad a(t) \to 0^+.$$

Vacuum solutions

- Vacuum solutions are solutions with $\rho(t) = n(t) = 0$ for all times.
- May act as future/past attractors of general solutions of the system of equations.
- Since $\dot{\phi} = 0$ when $\rho = n = 0$, vacuum solutions correspond to maximally symmetric vacuum spacetimes with cosmological constant $\phi = \Lambda$.
- The scale factor of vacuum solutions satisfies

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \frac{k}{a^2}, \quad a(0) = a_0, \quad \dot{a}(0) = H_0 a_0, \quad H_0 = \pm\sqrt{\frac{\Lambda}{3} - \frac{k}{a_0^2}}.$$

  The explicit form of the solution depends on the values of the parameters k and Λ.

- Vaccum solutions are either de-Sitter (if $\Lambda > 0$ for all k), anti-de-Sitter (if $\Lambda < 0$ and $k = -1$), Minkowski (if $\Lambda = 0$, $k = 0$), or Milne (if $\Lambda = 0$, $k = -1$)

Vacuum matching solutions

- A remarkable difference with the diffusion-free case is the possibility that the energy density ρ vanished at some finite time t_0 while the scale factor a is still regular.

- This possibility arises because $\rho = 0$ is not a solution.

- By a time translation we may set $t_0 = 0$ for any given such type of solution and therefore we assume that

$$\rho(0) = 0, \quad a(0) > 0, \quad |H(0)| < \infty.$$

- Since $\dot{\rho}(0) > 0$, hence $\rho(t) < 0$ for $t < 0$.

- To avoid this unphysical region of spacetime, we replace it with suitable vaccum solutions.

- This extended spacetime is singularity free in the past and in the region $t \leq 0$ the scale factor is given by the vacuum solutions depending on the value of k, $\phi(0) = \Lambda$ and $H_0 = H(0)$.

- The matching at $t = 0$ is C^2 in the scale factor $a(t)$ and C^0 in the scalar field $\phi(t)$ and in the energy density $\rho(t)$.

Qualitative dynamics

- We introduce

$$D = \sqrt{H^2 + \frac{1}{a^2}}$$

and a new time variable τ by

$$\frac{d}{d\tau}(\cdot) = \frac{1}{D}\frac{d}{dt}(\cdot).$$

In the following we use the notation $(\cdot)' = \frac{d}{d\tau}(\cdot)$.

- Dimensionless variables

$$H_D = \frac{H}{D}, \quad M_D = \frac{1}{aD}, \quad \Omega_D = \frac{\rho}{3D^2}, \quad Y_D = \frac{\phi}{3D^2}, \quad X_D = \frac{\dot{\phi}}{3D^3}.$$

- These variables satisfy the algebraic constraints

$$H_D^2 + M_D^2 = 1, \quad X_D + \beta M_D^3 = 0, \quad H_D^2 = \Omega_D + Y_D - kM_D^2,$$

where we recall that $\beta = \frac{\sigma n_0 a_0^3}{3}$.

- The evolution equation for the dimensional variable D is given by

$$D' = -H_D D(1 + qH_D^2),$$

- while the remaining variables satisfy

$$\begin{aligned} H_D' &= -qH_D^2(1 - H_D^2), \\ M_D' &= qH_D^3 M_D, \\ \Omega_D' &= -X_D + 2H_D\Omega_D(1 + qH_D^2 - \frac{3}{2}\gamma), \\ Y_D' &= X_D + 2H_D Y_D(1 + qH_D^2), \\ X_D' &= 3qH_D^3 X_D, \end{aligned}$$

- where the deceleration parameter

$$qH_D^2 = -Y_D + \Big(\frac{3}{2}\gamma - 1\Big)\Omega_D.$$

- Note that the equation on D decouples from the rest of the system; this is due to the fact that D is dimensional, while the other variables have no physical dimension (in our units).

- From the constraint equations, we obtain a reduced 2-dimensional dynamical system.
- We choose to work with the variables (Y_D, H_D).
- The reduced dynamical system is then given by

$$H'_D = \left[Y_D + (1 - \frac{3}{2}\gamma)\Omega_D\right](1 - H_D^2),$$
$$Y'_D = -\beta(1 - H_D^2)^{\frac{3}{2}} + 2H_D Y_D\left[1 - Y_D - (1 - \frac{3}{2}\gamma)\Omega_D\right],$$

where

$$\Omega_D = H_D^2 + k(1 - H_D^2) - Y_D.$$

- As opposed to the standard diffusion-free case ($\beta = 0$), the variable Ω_D is not bounded and $\Omega_D = 0$ is not an invariant boundary.
- In fact

$$(\Omega'_D)_{|\Omega_D=0} = -X_D = \beta(1 - H_D^2)^{3/2} > 0,$$

and the curve $\Omega_D = 0$ acts as "semipermeable membrane": *the flow can cross this line only in one direction*.

- In particular, the region $\Omega_D > 0$ is future invariant and if an orbit in the region $\Omega_D > 0$ intersects the vacuum line $\Omega_D = 0$ in the past, then Ω_D is negative for all earlier times along this orbit.

- The solutions of the original system corresponding to these orbits can be matched to a suitable vacuum solution at the time when the boundary Ω_D is crossed.

- By the preceding remarks, we only need to worry about the region $\Omega_D > 0$, which in terms of the variables (Y_D, H_D) means

$$\Omega_D > 0 \Leftrightarrow Y_D < H_D^2 + k(1 - H_D^2).$$

- The state-space $\mathcal{X}$ for the reduced dynamical system is then

$$\mathcal{X} = \{(Y_D, H_D) \in \mathbb{R} \times (-1, 1) \ : \ \Omega_D > 0\}.$$

- Our next goal is to study the qualitative behavior of the flow of the dynamical system and to present the physical interpretation of this analysis in terms of solutions of the Einstein equations and their asymptotic behavior.

Fixed Points

- The dynamical system possesses five fixed points, four of which are located on the boundary ($H_D = \pm 1$) and one in the interior.

Fixed point	$\mathbf{Y_D}$	$\mathbf{H_D}$	H(t)
dS_-	1	-1	$-c$
dS_+	1	1	c
F_-	0	-1	$\frac{2\gamma}{3} t_-^{-1}$
F_+	0	1	$\frac{2\gamma}{3} t_+^{-1}$
S_k	$(1 - \frac{2}{3\gamma})(k + (1-k)\frac{\delta_k^2}{1+\delta_k^2})$	$\frac{\delta_k}{\sqrt{1+\delta_k^2}}$	$\tilde{c} t^{-1}$

Table: Fixed points of the dynamical system in the state-space $\overline{\mathcal{X}}$.

Flat case $k = 0$

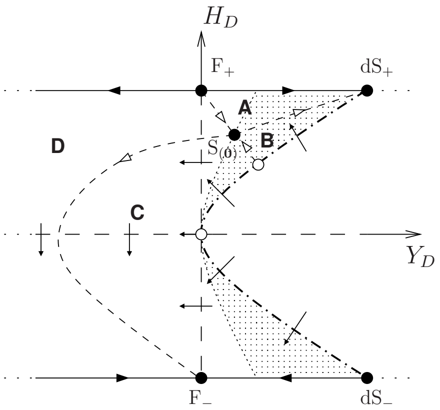
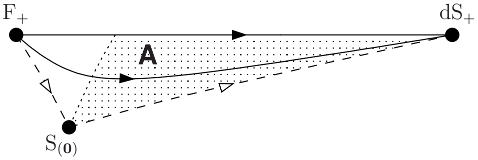


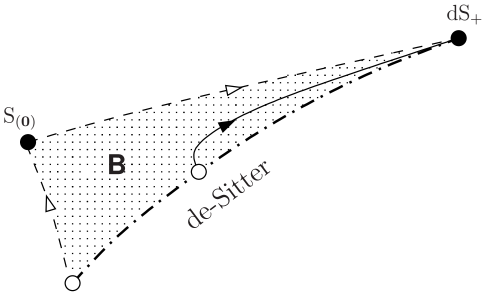
Figure 1: Partition of the state-space for $k = 0$

Region **A** for $k = 0$



- All orbits in the region **A** originate from the fixed point F_+ and terminate at dS_+.
- The solutions have a BIG BANG singularity in the past, while in the future they are singularity free and asymptotically de-Sitter.
- Since $H_D > 0$ in region **A**, and all orbits enter the shadowed region before converging to dS_+, the solutions are forever expanding and, after some finite time, the expansion becomes accelerated.

Region **B** for $k = 0$

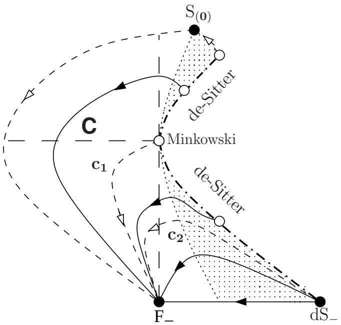


- All orbits in the region **B** converge in the future to the fixed point dS_+, while in the past they intersect the vacuum line $\Omega_D = 0$, where the scale factor can be continued to the de-Sitter solution for $k = 0$.

- The corresponding solutions of the Einstein equations are singularity free and asymptotically de-Sitter both in the past and future time directions.

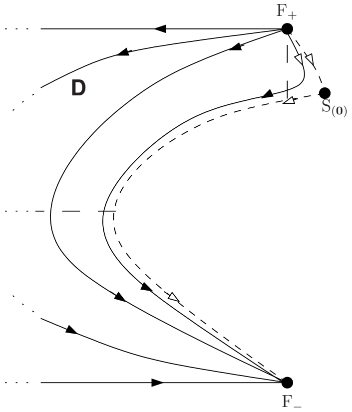
- Since $H_D > 0$ and the region **B** is completely shadowed, solutions corresponding to orbits in this region undergo accelerated expansion *for all times*.

Region **C** for $k = 0$



- Toward the future, orbits converge to the fixed point F_-. The corresponding solutions possess a BIG CRUNCH singularity toward the future.
- Toward the past, orbits in the region **C** intersect the vacuum line $\Omega_D = 0$.
- There are two special orbits c_1, c_2 that divide the region into three invariant regions.

Region **D** for $k = 0$



- All orbits originate from the fixed point F_+, and terminate at F_-.
- The solutions become singular at finite time in both time directions. BIG BANG into the past and BIG CRUNCH into the future.
- The solutions are initially expanding, until they reach a stage of maximum extension, and then they recollapse into the future singularity.

Open case $k = -1$

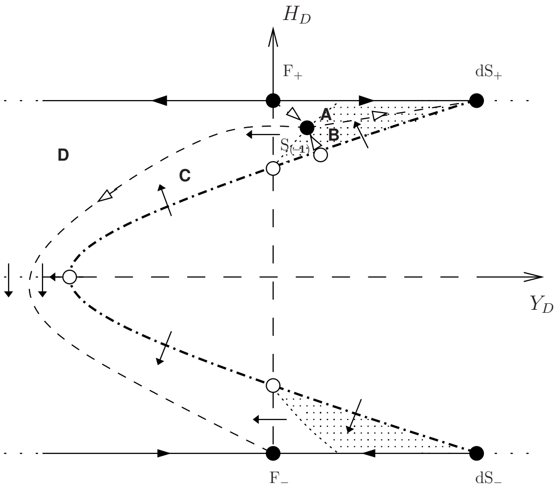
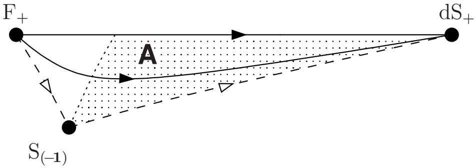
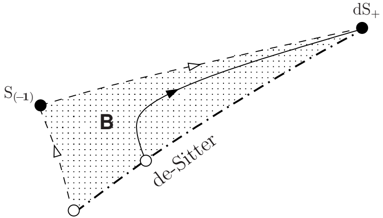


Figure 1: Partition of the state-space for $k = -1$

Region **A** and **B** for $k = -1$

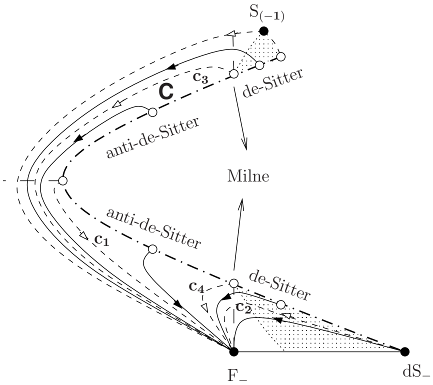


- The solutions are forever expanding having a BIG BANG singularity into the past are asymptotically de-Sitter into the future. After some finite time the solutions are expanding at an accelerated rate.



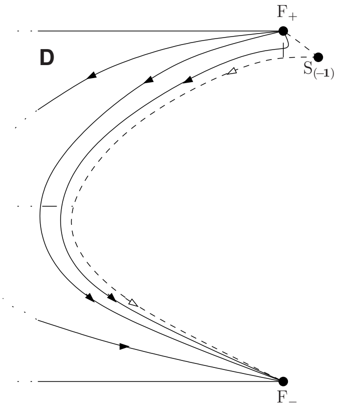
- The solutions are forever expanding with acceleration and are asymptotically de-Sitter into the future. At some finite time the solutions can be matched with expanding de-Sitter with $k = -1$ for all earlier times.

Region **C** for $k = -1$



- The matching of the scale factor at $\Omega_D = 0$ is either with de-Sitter ($Y_D > 0$), or with Milne ($Y_D = 0$), or with anti-de-Sitter ($Y_D < 0$).
- There are two special orbits, c_3 and c_4, intersecting the vacuum line in the expanding and contracting Milne solution.
  - Both these orbits converge to F_- in the future and divide the region **C** into three parts.

Region **D** for $k = -1$



- All orbits originate from the fixed point F_+, and terminate at F_-.
- The solutions become singular at finite time in both time directions. BIG BANG into the past and BIG CRUNCH into the future.

Closed case $k = +1$

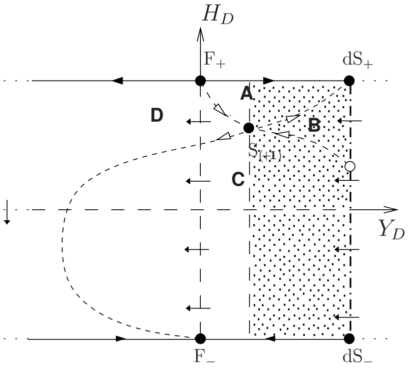
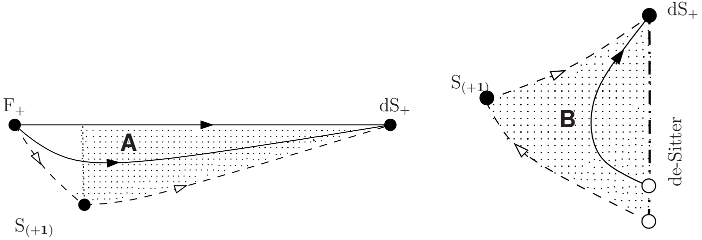


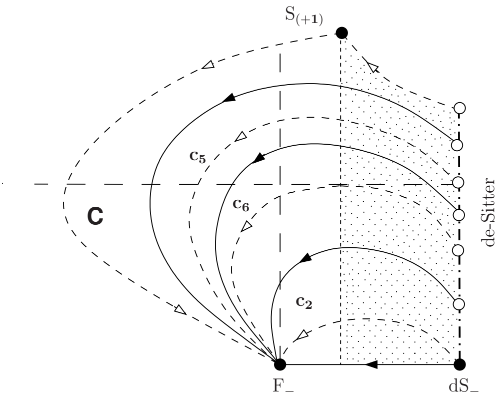
Figure 1: Partition of the state-space for $k = +1$

Regions **A** and **B** for $k = 1$



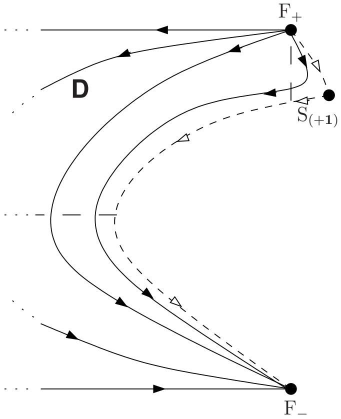
- The qualitative behavior of the orbits in regions **A** and **B** for this case is similar to the previous cases $k = 0$ and $k = -1$.

Region **C** for $k = 1$



- The matching of the scale factor at $\Omega_D = 0$ is with de-Sitter ($k = +1$).
- There are two special orbits, c_5 and c_6, intersecting the vacuum line in the expanding and contracting de-Sitter solutions, respectively.
  - Both these orbits converge to F_- in the future and divide the region **C** into three parts.
  - The orbits above c_5 are initially expanding and then collapsing.
  - Between c_5 and c_6 the solutions are initially contracting $\rightarrow$ expanding $\rightarrow$ recollapse.
  - The orbits below c_6 are forever contracting.

Region **C** for $k = 1$



- All orbits originate from the fixed point F_+, and terminate at F_-.
- The solutions become singular at finite time in both time directions. BIG BANG into the past and BIG CRUNCH into the future.

Conclusions and further work

- Conclusions:
  - In this work we considered a cosmological model based on the Einstein equations with cosmological scalar field and with fluid matter source.
  - The scalar field can be viewed as a background medium in which the fluid particles undergo diffusion.
  - We take spacetime to have a Robertson-Walker line element, so that the model studied here is spatially homogeneous and isotropic.
  - The matter field variables are solutions of a non linear system of ordinary differential equations.
  - We were able to obtain all solutions in which the scale factor is linear on time.
  - In order to understand the dynamical properties of general solutions, we rewrite the system in terms of normalized dynamical variables.
  - We have shown that typical solutions of the Einstein equations can be classified according to their past and future asymptotic behavior into four classes, which we called **A**, **B**, **C**, **D**.
  - In particular, solutions of type **B** describe a singularity-free spacetime which is forever expanding with acceleration and which is asymptotic to de-Sitter spacetime at both early and late times.
- Future work:
  - Use these methods to study more general anisotropic models of Bianchi type
  - Among inhomogeneous models, spherical symmetry where the scalar fiel can mimic dark matter through diffusion