# Vacuum polarization on the brane

Elizabeth Winstanley

Consortium for Fundamental Physics
School of Mathematics and Statistics
The University of Sheffield

Breen, Hewitt, Ottewill and EW, *Physical Review* **D 92** 084039 (2015)

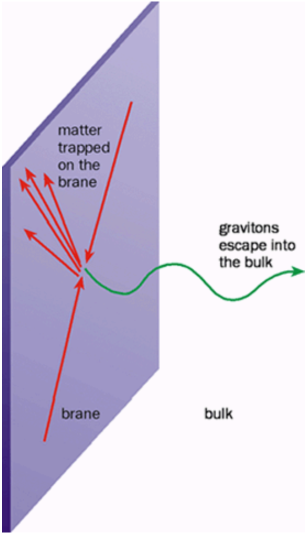
# Outline

# AADD brane-world scenario

- Four-dimensional brane where all Standard Model particles live
- $D$-dimensional bulk
- Only gravitons propagate in the bulk

$$M_P^2 \sim R^{D-4} M_*^{D-2}$$

[ Antoniadis, Arkani-Hamed, Dimopoulos and Dvali, hep-ph/9803315 ; hep-ph/9804398 ]



# Physics in the AADD brane-world scenario

- Fundamental, higher-dimensional scale of quantum gravity may be as low as the TeV scale
- Collider experiment with centre-of-mass energy $\sqrt{s} > M_*$ will probe strong-gravity regime
- Creation of microscopic black holes?

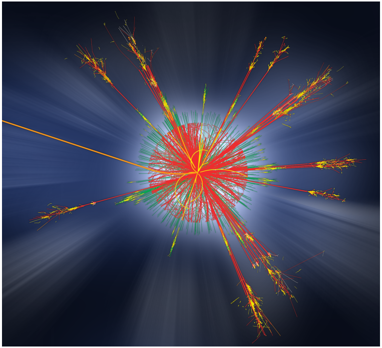
[ Banks and Fischler, `hep-th/9906038` ]

[ Images: ATLAS Experiment ©2014 CERN ; Simon Swordy/University of Chicago, NASA ]

# Hawking radiation from brane black holes

## QFT in curved space-time

- Classical metric
- Quantum field propagating on this background
- Black hole emits Hawking radiation at temperature $T_H$
- Fluxes can be computed without renormalization



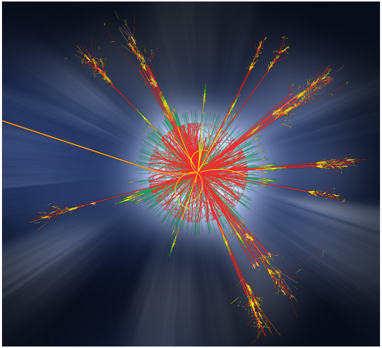
[ Image: ATLAS Experiment ©2014 CERN ]

# Hawking radiation from brane black holes

## QFT in curved space-time

- Classical metric
- Quantum field propagating on this background
- Black hole emits Hawking radiation at temperature $T_H$
- Fluxes can be computed without renormalization

What about quantities which require renormalization?



[ Image: ATLAS Experiment ©2014 CERN ]

# Vacuum polarization

## Quantum scalar field

- Massless conformally coupled quantum scalar field $\hat{\phi}$
- Hawking radiation in the bulk suppressed relative to the brane
  [ Casals et al, arXiv:0801.4910 [hep-th] ; Emparan et al, hep-th/0003118 ; Harris and Kanti, hep-ph/0309054 ]
- On the brane

$$\left[ \Box - \frac{1}{6}\mathcal{R} \right] \phi = 0$$

# Vacuum polarization

## Quantum scalar field

- Massless conformally coupled quantum scalar field $\hat{\phi}$
- Hawking radiation in the bulk suppressed relative to the brane
  [ Casals et al, arXiv:0801.4910 [hep-th] ; Emparan et al, hep-th/0003118 ; Harris and Kanti, hep-ph/0309054 ]
- On the brane

$$\left[ \Box - \frac{1}{6}\mathcal{R} \right] \phi = 0$$

## Vacuum polarization $\langle \hat{\phi}^2 \rangle$

- Simplest non-trivial expectation value
- Precursor to a computation of $\langle \hat{T}_{\mu\nu} \rangle$

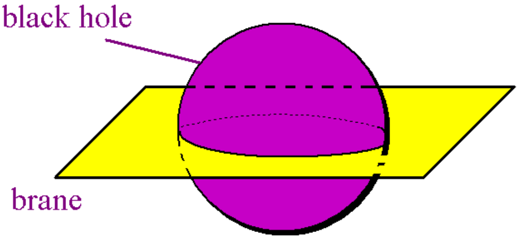
$$G_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle$$

# Brane-world black hole geometry

## $D$-dimensional Schwarzschild-Tangherlini black hole

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}dr^2 + r^2\,d\Omega_{D-2}$$

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^{D-3}$$

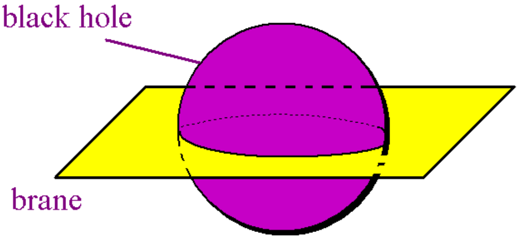


# Brane-world black hole geometry

## $D$-dimensional Schwarzschild-Tangherlini black hole

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}dr^2 + r^2\,d\Omega_{D-2}$$

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^{D-3}$$

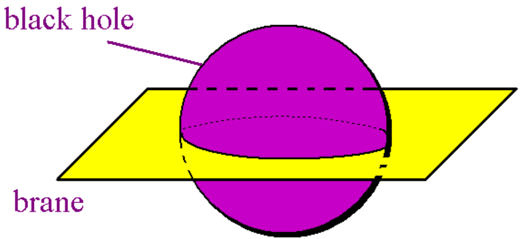


# Brane-world black hole geometry

## Metric on the brane

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}dr^2 + r^2\,d\Omega_2$$

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^{D-3}$$

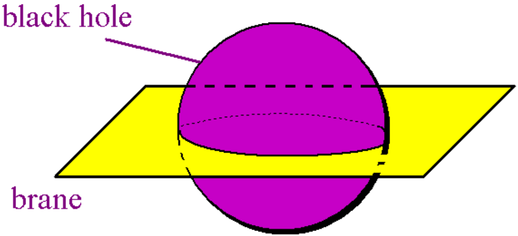


# Brane-world black hole geometry

## Metric on the brane

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}dr^2 + r^2\,d\Omega_2$$

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^{D-3}$$

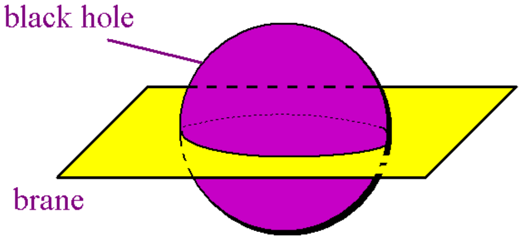


# Brane-world black hole geometry

## Euclideanized space-time metric $t \to i\tau$

$$ds^2 = f(r)\, d\tau^2 + f(r)^{-1} dr^2 + r^2\, d\Omega_2$$

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^{D-3}$$



# Quantum state

## Hawking temperature

$$T_H = \frac{D-3}{4\pi r_h}$$

# Quantum state

## Hawking temperature

$$T_H = \frac{D-3}{4\pi r_h}$$

## Hartle-Hawking state $|H\rangle$

- Black hole in thermal equilibrium with a heat bath at temperature $T_H$
- Quantum state with the most symmetries - easiest for computations
- Differences in expectation values between two quantum states do not require renormalization

# Point-split Euclidean Green's function $G_E(x;x')$

$$\langle \hat{\phi}^2 \rangle_{\text{unren}} = \Re e \left[ \lim_{x' \to x} G_E(x;x') \right]$$

$$\left[ \Box - \frac{1}{6}\mathcal{R} \right] G_E(x;x') = -g^{-\frac{1}{2}}(x)\delta^4(x-x')$$

# Point-split Euclidean Green's function $G_E(x;x')$

## Mode sum

$$G_E(x;x') = \frac{T_H}{4\pi} \sum_{n=-\infty}^{\infty} \exp\left[i\omega(\tau - \tau')\right] \times \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\gamma) C_{\omega\ell}\, p_{\omega\ell}(r_<) q_{\omega\ell}(r_>)$$

# Point-split Euclidean Green's function $G_E(x;x')$

## Mode sum

$$G_E(x;x') = \frac{T_H}{4\pi} \sum_{n=-\infty}^{\infty} \exp\left[i\omega(\tau-\tau')\right] \times \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\gamma) C_{\omega\ell} p_{\omega\ell}(r_<) q_{\omega\ell}(r_>)$$

$T_H$ - Hawking temperature

# Point-split Euclidean Green's function $G_E(x;x')$

## Mode sum

$$G_E(x;x') = \frac{T_H}{4\pi} \sum_{n=-\infty}^{\infty} \exp\left[i\omega(\tau-\tau')\right] \times \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\gamma) C_{\omega\ell} p_{\omega\ell}(r_<) q_{\omega\ell}(r_>)$$

$$\omega = 2\pi n T_H$$

# Point-split Euclidean Green's function $G_E(x;x')$

## Mode sum

$$
\begin{aligned}
G_E(x;x') &= \frac{T_H}{4\pi} \sum_{n=-\infty}^{\infty} \exp\left[i\omega(\tau-\tau')\right] \\
&\quad \times \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\gamma) C_{\omega\ell}\, p_{\omega\ell}(r_<) q_{\omega\ell}(r_>)
\end{aligned}
$$

$P_\ell(\cos\gamma)$ - Legendre polynomial
$\gamma$ - angular separation

# Point-split Euclidean Green's function $G_E(x;x')$

## Mode sum

$$G_E(x;x') = \frac{T_H}{4\pi} \sum_{n=-\infty}^{\infty} \exp\left[i\omega(\tau-\tau')\right] \times \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\gamma) C_{\omega\ell}\, p_{\omega\ell}(r_<) q_{\omega\ell}(r_>)$$

$C_{\omega\ell}$ - normalization constant

# Point-split Euclidean Green's function $G_E(x;x')$

## Mode sum

$$G_E(x;x') = \frac{T_H}{4\pi} \sum_{n=-\infty}^{\infty} \exp\left[i\omega(\tau-\tau')\right] \times \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\gamma) C_{\omega\ell} p_{\omega\ell}(r_<) q_{\omega\ell}(r_>)$$

$p_{\omega\ell}(r)$ and $q_{\omega\ell}(r)$ satisfy the radial equation

$$0 = f\frac{d^2S}{dr^2} + \left(\frac{2f}{r} + \frac{df}{dr}\right)\frac{dS}{dr} - \left[\frac{\omega^2}{f} + \frac{\ell(\ell+1)}{r^2} + \frac{\mathcal{R}}{6}\right] S$$

$$r_< = \min\{r, r'\} \qquad r_> = \max\{r, r'\}$$

# Renormalizing the vacuum polarization

$$\langle \hat{\phi}^2 \rangle_{\text{unren}} = \Re e \left[ \lim_{x' \to x} G_E(x; x') \right]$$

# Renormalizing the vacuum polarization

$$\langle \hat{\phi}^2 \rangle_{\text{unren}} = \Re e \left[ \lim_{x' \to x} G_E(x;x') \right]$$

$G_E(x;x')$ diverges as $x' \to x$

# Renormalizing the vacuum polarization

$$\langle \hat{\phi}^2 \rangle_{\text{unren}} = \Re e \left[ \lim_{x' \to x} G_E(x;x') \right]$$

$G_E(x;x')$ diverges as $x' \to x$

$$\langle \hat{\phi}^2 \rangle_{\text{div}} = \frac{1}{8\pi^2 \sigma} + \frac{1}{96\pi^2} \frac{\mathcal{R}_{\alpha\beta}\sigma^\alpha \sigma^\beta}{\sigma}$$

$2\sigma$ - square of geodesic distance between $x$, $x'$
$\sigma^\alpha = \sigma^{;\alpha}$

# Renormalizing the vacuum polarization

$$\langle \hat{\phi}^2 \rangle_{\text{unren}} = \Re e \left[ \lim_{x' \to x} G_E(x; x') \right]$$

$G_E(x; x')$ diverges as $x' \to x$

$$\langle \hat{\phi}^2 \rangle_{\text{div}} = \frac{1}{8\pi^2 \sigma} + \frac{1}{96\pi^2} \frac{\mathcal{R}_{\alpha\beta} \sigma^\alpha \sigma^\beta}{\sigma}$$

$2\sigma$ - square of geodesic distance between $x$, $x'$
$\sigma^\alpha = \sigma^{;\alpha}$

## Renormalized vacuum polarization

$$\langle \hat{\phi}^2 \rangle_{\text{ren}} = \Re e \left\{ \lim_{x' \to x} \left[ G_E(x; x') - \langle \hat{\phi}^2 \rangle_{\text{div}} \right] \right\}$$

# $\langle \hat{\phi}^2 \rangle_{\text{ren}}$ outside the event horizon $r > r_h$

## Time-like point-splitting

$$\langle \hat{\phi}^2 \rangle_{\text{ren}} = \langle \hat{\phi}^2 \rangle_{\text{analytic}} + \langle \hat{\phi}^2 \rangle_{\text{numeric}}$$

$$\langle \hat{\phi}^2 \rangle_{\text{analytic}} = \frac{T_H^2}{12f} - \frac{1}{192\pi^2 f}\left(\frac{df}{dr}\right)^2 + \frac{1}{96\pi^2}\frac{d^2 f}{dr^2} + \frac{1}{48\pi^2 r}\frac{df}{dr}$$

$$\langle \hat{\phi}^2 \rangle_{\text{numeric}} = \frac{T_H}{4\pi}\sum_{\ell=0}^{\infty}\left[(2\ell+1)\,C_{0\ell}p_{0\ell}(r)q_{0\ell}(r) - \frac{1}{r\sqrt{f}}\right]$$
$$+\frac{T_H}{2\pi}\sum_{n=1}^{\infty}\left\{\sum_{\ell=0}^{\infty}\left[(2\ell+1)\,C_{\omega\ell}p_{\omega\ell}(r)q_{\omega\ell}(r) - \frac{1}{r\sqrt{f}}\right] + \frac{\omega}{f}\right\}$$

[ Anderson, Hiscock and Samuel, *PRD* **51** 4337 (1995) ;
EW and Young, `arXiv:0708.3820 [gr-qc]` ]

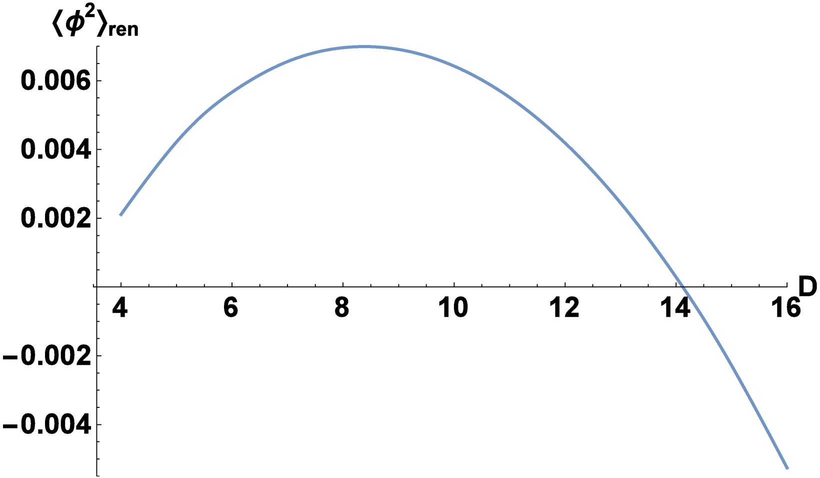
# $\langle \hat{\phi}^2 \rangle_{\text{ren}}$ on the event horizon $r = r_h = 1$

## Radial point-splitting $\langle \hat{\phi}^2 \rangle_{\text{ren}}$

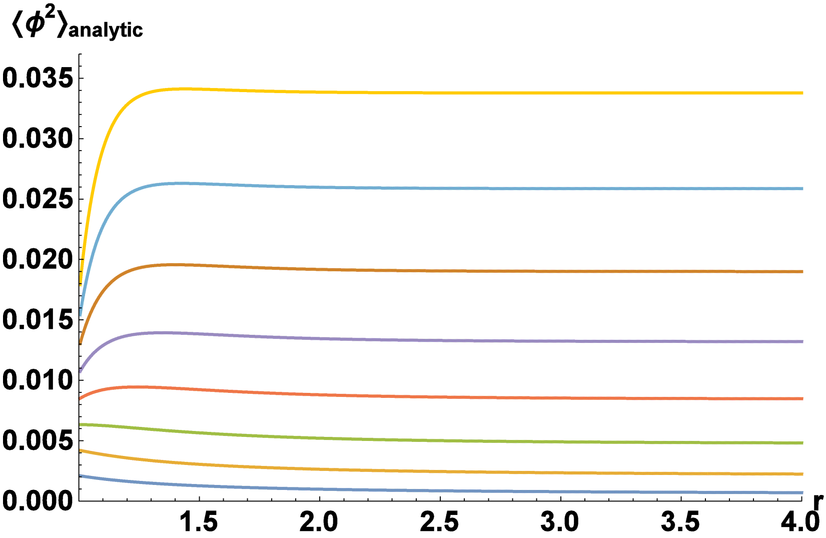
$$\frac{D-3}{48\pi^2} + \frac{(D-4)(D-5)}{96\pi^2}\left[\psi\left(\frac{D-2}{2} - \frac{\sqrt{(D-4)(D-2)}}{2\sqrt{3}}\right)\right.$$
$$\left. + \psi\left(\frac{D-2}{2} + \frac{\sqrt{(D-4)(D-2)}}{2\sqrt{3}}\right) - 2\ln(D-3)\right]$$
$$+ \frac{1}{16\pi^2(D-3)}\sum_{j=0}^{D-4} j(j-D+4)\left[\psi\left(\frac{6j+3(D-2)-\sqrt{3(D-4)(D-2)}}{6(D-3)}\right)\right.$$
$$\left. + \psi\left(\frac{6j+3(D-2)+\sqrt{3(D-4)(D-2)}}{6(D-3)}\right)\right].$$

[ Breen and Ottewill, `arXiv:1111.3298 [gr-qc]` ]

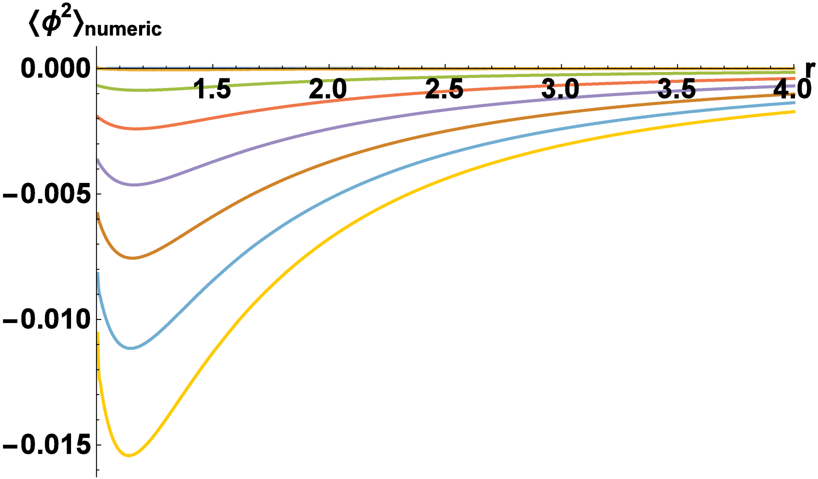
# $\langle \hat{\phi}^2 \rangle_{\rm ren}$ on the event horizon $\qquad r_h = 1$



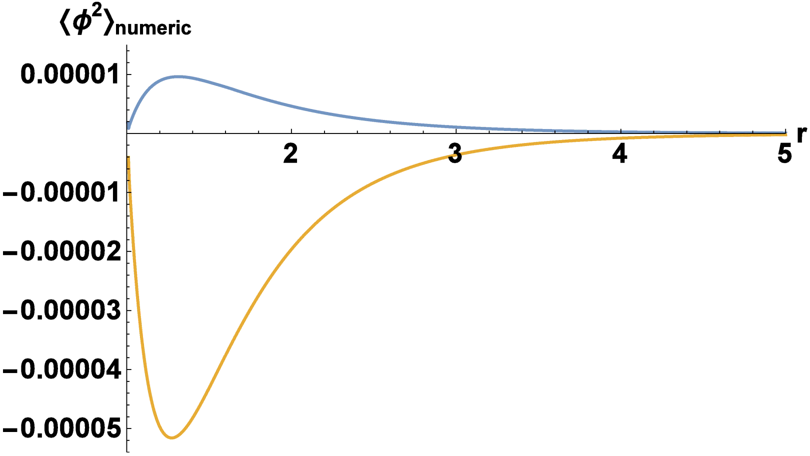
## $\langle\hat{\phi}^2\rangle_{\text{analytic}}$ $r_h = 1$



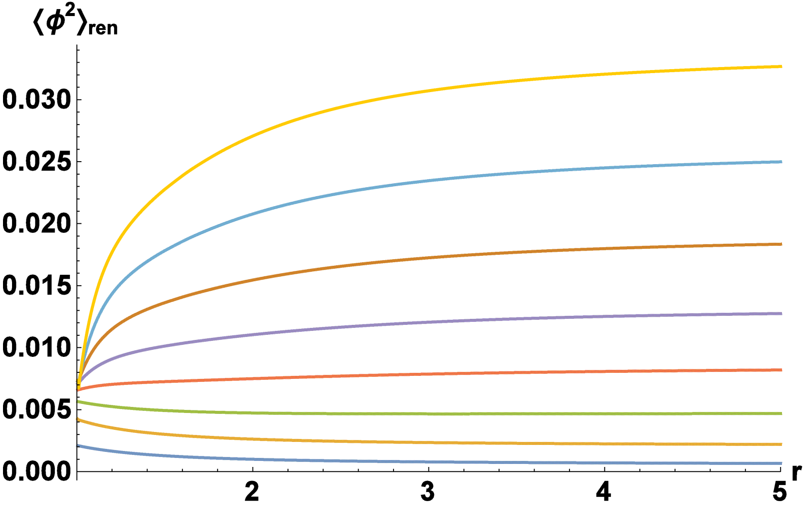
$\langle \hat{\phi}^2 \rangle_{\text{numeric}}$ $\qquad r_h = 1$



## $\langle \hat{\phi}^2 \rangle_{\text{numeric}}$ $r_h = 1$



$$\langle\hat{\phi}^2\rangle_{\text{ren}} = \langle\hat{\phi}^2\rangle_{\text{analytic}} + \langle\hat{\phi}^2\rangle_{\text{numeric}} \qquad r_h = 1$$



# Summary

- Static, spherically symmetric, $D$-dimensional Schwarzschild-Tangherlini black hole
- Quantum scalar field $\hat{\phi}$ on the four-dimensional brane
- Renormalized vacuum polarization $\langle \hat{\phi}^2 \rangle_{\rm ren}$

# Summary

- Static, spherically symmetric, $D$-dimensional Schwarzschild-Tangherlini black hole
- Quantum scalar field $\hat{\phi}$ on the four-dimensional brane
- Renormalized vacuum polarization $\langle \hat{\phi}^2 \rangle_{\text{ren}}$

## Vacuum polarization in the Hartle-Hawking state

- On the horizon
  - Positive for $D = 4, \ldots, 14$
  - Negative for $D > 14$
- As $r \to \infty$

$$\langle \hat{\phi}^2 \rangle_{\text{ren}} \Big|_{r \to \infty} = \frac{(D-3)^2}{192\pi^2} = \frac{T_H^2}{12}$$